

A FOOTNOTE ON EXPANDING MAPS

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ABSTRACT. I introduce Banach spaces on which it is possible to precisely characterize the spectrum of the transfer operator associated to a piecewise expanding map with Hölder weight.

1. INTRODUCTION

Lately there is some renewed interest in different norms which allow one to analyze transfer operators associated to expanding maps. Such an interest has several motivations, one of the most relevant being the study of semi flows arising from Lorentz like models. At the same time, the extension of transfer operator methods to the hyperbolic setting [7, 11, 12, 5, 6, 2, 3, 10, 18, 22, 4], just to mention a few, has revitalized the subject. Particular attention has been devoted to the case in which the map is piecewise smooth and its derivative or the weight have low regularity (Hölder instead of \mathcal{C}^1). Several possibilities have been and are currently being explored trying to improve on the classical BV scheme [20, 9, 17] or its relevant variants [14, 19]. Two recent interesting contributions are [21, 8].

The purpose of this note is to comment on an old proposal of mine, put forward in footnote 12 of [16, page 193], that has gone mostly unnoticed and/or not understood. Here I show that it can be easily applied to many relevant situations yielding the strongest results so far. I present the approach in the expanding one dimensional case but I see no obstacles in extending it to higher dimensions (following [17]) or, with some more work, to the hyperbolic setting.

In the next section I will detail the proposed Banach space which is a weakening of BV in the spirit of fraction order Sobolev spaces but avoiding the use of Fourier transforms altogether. In the final section I will detail some settings where the above strategy can be applied and I will give the main result of the paper.

2. THE BANACH SPACE

Let \mathfrak{M} be the Banach space of complex valued Borel measures on $[0, 1]$ equipped with the total variation norm. For each $\varphi \in \mathcal{C}^1([0, 1], \mathbb{C})$, $\mu \in \mathfrak{M}$, let us define, for

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each $\alpha \in [0, 1]$,

$$|\varphi|_\alpha = \sup_{x \in [0, 1]} |\varphi(x)| + \sup_{x, y \in [0, 1]} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}$$

$$\|\mu\|_\alpha = \sup_{\{\varphi \in \mathcal{C}^1 : |\varphi|_\alpha \leq 1\}} |\mu(\varphi')|.$$

We can then define $\mathcal{B}_\alpha = \{\mu \in \mathfrak{M} : \|\mu\|_\alpha < \infty\}$. Note that $\mathcal{B}_0 = BV$ and $\mathcal{B}_1 = \mathfrak{M}$.

Remark 2.1. *In the following we give, for the reader's convenience, a self contained proof of the relevant properties of the spaces \mathcal{B}_α . Note however that some results are known in larger generality than needed here [13, 23]. Also there is a connection between our spaces and the theory of BV functions on snowflake spaces [23].*

Lemma 2.2. *If $\alpha \in [0, 1)$ and $\mu \in \mathcal{B}_\alpha$, then μ is absolutely continuous with respect to Lebesgue and, calling h its density,¹*

$$|h|_{L^{\frac{1}{1-\alpha}}} \leq 2\|\mu\|_\alpha.$$

In addition \mathcal{B}_α is a Banach space.

Proof. Let $\varphi \in \mathcal{C}^0$ and $\mu \in \mathcal{B}_\alpha$, and define $\phi(x) = \int_0^x \varphi$. Then

$$\mu(\varphi) = \mu(\phi').$$

Since $|\phi(x)| \leq |\varphi|_{L^{\frac{1}{1-\alpha}}}$ and

$$|\phi(x) - \phi(y)| = \left| \int_x^y \varphi \right| \leq |\varphi|_{L^{\frac{1}{1-\alpha}}} |x - y|^\alpha,$$

it follows that $|\phi|_\alpha \leq 2|\varphi|_{L^{\frac{1}{1-\alpha}}}$ hence

$$|\mu(\varphi)| \leq 2\|\mu\|_\alpha |\varphi|_{L^{\frac{1}{1-\alpha}}}.$$

Thus μ belongs to the dual of $L^{\frac{1}{1-\alpha}}$, i.e. $L^{\frac{1}{\alpha}}$. To verify that \mathcal{B}_α is a Banach space it suffices to see that it is complete. Let $\{\mu_n\} \subset \mathcal{B}_\alpha$ be a Cauchy sequence and $\{h_n\}$ be the respective densities. Then $\{h_n\}$ is a Cauchy sequence in $L^{\frac{1}{\alpha}}$, let h be its limit. Setting $\mu(\varphi) := \int_0^1 h\varphi$,²

$$|\mu(\varphi')| = \lim_{n \rightarrow \infty} |\mu_n(\varphi')| \leq C_\# |\varphi|_\alpha.$$

Thus, $\mu \in \mathcal{B}_\alpha$. □

Given the above Lemma we can as well consider the space of densities equipped with the norm

$$\|h\|_\alpha = \sup_{|\varphi|_\alpha \leq 1} \left| \int_0^1 h\varphi' \right|.$$

By a little abuse of notations we will call such a Banach space \mathcal{B}_α as well. Since $\mathcal{B}_\alpha \subset L^1([0, 1])$ it is then convenient to use L^1 rather than \mathfrak{M} as a weak space.³

Lemma 2.3. *For each $\alpha \in (0, 1)$ the unit ball of \mathcal{B}_α is relatively compact in $L^1([0, 1])$.*

¹ In this note the L^p spaces are all w.r.t. the Lebesgue measure.

² In this note I use $C_\#$ to designate a generic constant.

³ Note that the norm is exactly the same.

Proof. Note that the functions in the set $\{\int_0^x \varphi\}_{|\varphi|_\infty \leq 1}$ are uniformly Lipschitz and hence, by Ascoli-Arzelá, relatively compact in the α -Hölder functions. Thus, for each $\varepsilon > 0$ there exists a set $S_\varepsilon := \{\phi_i\}_{i=1}^{n_\varepsilon}$, $|\phi_i|_{C^1} < \infty$, such that, for each $\varphi \in L^\infty$, $|\varphi|_\infty \leq 1$,

$$\int h\varphi \leq \varepsilon \|h\|_\alpha + \sum_{i=1}^{n_\varepsilon} \left| \int h\phi'_i \right|.$$

To conclude, let $\overline{S}_k = \cup_{j \leq k} S_{2^{-j}}$. Since, for each $k \in \mathbb{N}$,

$$\sup_{\substack{\phi \in \overline{S}_k \\ n \in \mathbb{N}}} \left| \int h_n \phi' \right| < \infty$$

by Bolzano-Weierstrass we can extract a subsequence $\{n_j^k\}$ such that $\int h_{n_j^k} \phi'$ is convergent, when $j \rightarrow \infty$, for all $\phi \in \overline{S}_k$. The results follows then by the usual diagonalizing trick. \square

Lemma 2.4. *For each $\alpha \in (0, 1)$, $h \in \mathcal{B}_\alpha$ and φ Lipschitz we have⁴*

$$\int_0^1 h\varphi' \leq |\varphi|_\alpha \|h\|_\alpha.$$

Proof. It is convenient to extend h to be zero and φ to be continuous and constant outside $[0, 1]$, so we can regard all the integral as integral on \mathbb{R} . Let j_ε be a smooth mollifier, then⁵

$$\int_0^1 h\varphi' = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} (j_\varepsilon * h) \varphi' = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} h(j_\varepsilon * \varphi)' \leq \lim_{\varepsilon \rightarrow 0} \|h\|_\alpha |j_\varepsilon * \varphi|_\alpha \leq \|h\|_\alpha |\varphi|_\alpha.$$

\square

3. PIECEWISE DIFFERENTIABLE MAPS AND HÖLDER CONTINUOUS WEIGHTS

Let f be a piecewise expanding map of the interval $[0, 1]$ with smoothness (open) partition \mathcal{P} (possibly infinite), and let $\xi : [0, 1] \rightarrow \mathbb{C}$ a function that we will call be the *weight*. By *expanding* I mean that there exists $\lambda > 1$ such that $|f'| \geq \lambda$, where it is defined. Assume that, for each $p \in \mathcal{P}$, $\xi \in C^\beta(\overline{p}, \mathbb{R})$, with uniform β -Hölder constant, and $f, \frac{1}{f'} \in \mathcal{C}^0(\overline{p}, \mathbb{R})$. In addition, I require that $f' \in L^r$, for some $r \geq 1$, $\xi f' \in L^\infty$ and that there exists $\gamma \in [0, 1)$ such that

$$(3.1) \quad \sum_{p \in \mathcal{P}} \sup_{z \in p} |\xi(z) f'(x)^\gamma|^{\frac{1}{1-\gamma}} < \infty.$$

Next, let \mathcal{L}_ξ be the transfer operator defined by

$$\mathcal{L}_\xi h(x) = \sum_{y \in f^{-1}(x)} \xi(y) h(y).$$

The main result of this note is the following.

Theorem 3.1. *In the above setting, if $\beta > \frac{1}{r+1}$, then, for each $1 > \alpha > \max\{\gamma, 1 - \beta\}$, the spectral radius of \mathcal{L}_ξ , when acting on \mathcal{B}_α , is bounded from above by $|\xi f'|_\infty$, while the essential spectral radius is bounded by $8|\xi(f')^\alpha|_\infty$.*

⁴ Note that, by Rademacher's Theorem, φ is almost surely differentiable with bounded derivative, hence the integral is meaningful.

⁵ As usual $j * h(x) = \int_{\mathbb{R}} j(x-y) h(y) dy$.

Remark 3.2. Note that, as stated, in the case $\gamma = 0, \beta = 1$ the Lemma does not cover the case $\alpha = 0$. This is the usual BV case and it is well known already. Also, in the case $\beta = 0$ there is no reason to expect good spectral properties for \mathcal{L}_ξ .

Remark 3.3. As usual, by applying Theorem 3.1 to a large power of \mathcal{L}_ξ , much sharper estimates of the spectral and essential spectral radius can be obtained. I leave this exercise to the reader.

Remark 3.4. To compare the above result with the literature remark that [14], at least in the published version, applies only to the case $\xi = \frac{1}{f'}$ and when the partition \mathcal{P} is finite. The results in [21] apply only to the case $f' \in L^\infty$. Finally, in [8] it is assumed (3.1) with $\gamma = 0$. Note that

$$\sum_p |\xi(f')^\gamma|_{L^\infty(p)}^{\frac{1}{1-\gamma}} \leq |\xi f'|_{L^\infty}^{\frac{\gamma}{1-\gamma}} \sum_p |\xi|_{L^\infty(p)} \leq C_\# \sum_p |\xi|_{L^\infty(p)}$$

thus the present condition is weaker. Also in [8] it appears the condition $f' \in L^r$ with $\beta > \frac{1}{r}$ that here is replaced by $\beta > \frac{1}{r+1}$. In particular, if $\beta > \frac{1}{2}$, one can treat the case $r = 1$, which is the natural condition when the partition is finite.

Remark 3.5. The reader should be advised that the goal of this note is not to treat the most general case but to show that the \mathcal{B}_α spaces can be conveniently used to investigate a vast class of problems. The optimal conditions under which Theorem 3.1 holds depend heavily on the situation. The present treatment is specially adapted to the case of finite partitions with weights that can also be zero. In the case of infinite partitions it is probably more natural to consider weight of the form $\xi = e^\phi$, where ϕ is called the potential, and impose the Hölder condition on the potential, see Theorem 3.7.⁶

Theorem 3.1 follows in a standard way (see [1]) from Lemma 2.3 and the next Lasota-Yorke inequality.

Lemma 3.6. If $\beta > \frac{1}{r+1}$, then for each $1 > \alpha > \max\{\gamma, 1 - \beta\}$, there exists $B > 0$ such that, for all $h \in \mathcal{B}_\alpha$,

$$\begin{aligned} |\mathcal{L}h|_{L^1} &\leq |\xi f'|_\infty |h|_{L^1} \\ \|\mathcal{L}h\|_\alpha &\leq 8|\xi \cdot (f')^\alpha|_\infty \|h\|_\alpha + B|h|_{L^1}. \end{aligned}$$

Proof. Let $h \in \mathcal{B}_\alpha$. For each $\varphi \in \mathcal{C}^1$ such that $|\varphi|_\alpha \leq 1$, we have

$$\int \mathcal{L}_\xi h \cdot \varphi' = \sum_{p \in \mathcal{P}} \int_p h \xi(\varphi \circ f)'.$$

First of all we want to take care of the fact that f' may blow up at the boundaries of $p \in \mathcal{P}$ so that $\varphi \circ f$ may fail to be Lipschitz on \bar{p} . Note that if this is the case, since by hypothesis $|\xi(\varphi \circ f)'|_\infty \leq |\xi f'|_\infty \leq C_\#$, it follows that ξ is zero at such points. For each $\varepsilon > 0$ we can then consider the functions $\tilde{\xi}_\varepsilon(z) = \max\{|\xi(z)| - \varepsilon, 0\}$ and $\tilde{\xi}_\varepsilon = \frac{\xi}{|\xi|} \cdot \tilde{\xi}_\varepsilon$. Clearly $\tilde{\xi}_\varepsilon$ is zero in a neighborhood of the point in which f' explodes, also $|\tilde{\xi}_\varepsilon| \leq |\xi|$ and they have β -Hölder constant uniformly (in ε) proportional.

⁶ If ξ vanishes somewhere one can still use such a setting by introducing countably many artificial partition elements, in the spirit of billiards *homogeneity strips*.

Clearly for each h, φ there exists ε such that⁷

$$\left| \sum_{p \in \mathcal{P}} \int_p h(\xi - \tilde{\xi}_\varepsilon)(\varphi \circ f)' \right| \leq \frac{1}{2} \|h\|_\alpha.$$

Next, for each $k \in \mathbb{N}$, let \mathcal{P}_k be a refinement of \mathcal{P} such that all the elements of \mathcal{P} of length less than 2^{-k+1} are partitioned in elements of length between 2^{-k+1} and 2^{-k} . Let $\overline{\mathcal{P}}_k = \{p \in \mathcal{P}_k : p \notin \mathcal{P}\}$ and $\widehat{\mathcal{P}}_k = \{p \in \mathcal{P}_k : p \in \mathcal{P}\}$. For each $p \in \mathcal{P}_k$ let $x_p \in \overline{p}$ be such that $|\tilde{\xi}_\varepsilon(x_p)| = \inf_{z \in p} |\tilde{\xi}_\varepsilon(z)|$. For each $k \in \mathbb{N}$ let $\xi_k(x) = \tilde{\xi}_\varepsilon(x_p)$ for all $x \in p \in \mathcal{P}_k$. By hypothesis, $\xi_k \in L^\infty$ and

$$\|\xi_k - \tilde{\xi}_\varepsilon\|_\infty \leq C_\# 2^{-\beta k}.$$

It is now convenient to define $\rho_k = \xi_{k+1} - \xi_k$. Note that, for all $k_0 \in \mathbb{N}$,

$$\sum_{k \geq k_0} \rho_k = \tilde{\xi}_\varepsilon - \xi_{k_0},$$

where the convergence takes place in the uniform topology. Hence,⁸

$$\begin{aligned} \sum_{p \in \mathcal{P}_{k_0}} \int_p h \tilde{\xi}_\varepsilon(\varphi \circ f)' &= \sum_{k \geq k_0} \sum_{p \in \mathcal{P}_{k_0}} \int_p h(\varphi \circ f)' \rho_k + \sum_{p \in \mathcal{P}_{k_0}} \int_p h(\varphi \circ f)' \xi_{k_0} \\ (3.2) \quad &= \sum_{p \in \mathcal{P}_{k_0}} \int_p h(\varphi \circ f \cdot \xi_{k_0})' + \sum_{k \geq k_0} \int h \frac{d}{dx} \left[\int_0^x \sum_{p \in \mathcal{P}_{k_0}} \mathbf{1}_p(\varphi \circ f)' \rho_k \right]. \end{aligned}$$

To continue, let $\tilde{\ell}$ be linear in each $p \in \mathcal{P}_{k_0}$ and equal to $\varphi \circ f \cdot \xi_{k_0}$ on ∂p . Then $\eta_{k_0} = \varphi \circ f \cdot \xi_{k_0} - \tilde{\ell} \in \mathcal{C}^0$, in fact Lipschitz. In addition, for $x, y \in p \in \mathcal{P}$,

$$(3.3) \quad |\varphi \circ f(x) - \varphi \circ f(y)| \leq \left| \int_x^y f'(z) dz \right|^\alpha \leq |f'(\zeta)|^\alpha |x - y|^\alpha$$

for some $\zeta \in [x, y]$. Thus for each $x, y \in p \in \mathcal{P}_{k_0}$,

$$|\eta_{k_0}(x) - \eta_{k_0}(y)| \leq 2|\xi(f')|^\alpha_\infty |x - y|^\alpha.$$

On the other hand, if x and y belong to different elements of $p \in \mathcal{P}_{k_0}$, let $b_1, b_2 \in [x, y]$ the boundaries of the elements to which x and y belong, respectively. Since $\eta_{k_0} = 0$ at the boundaries of the elements of \mathcal{P}_{k_0} , we have

$$\begin{aligned} |\eta_{k_0}(x) - \eta_{k_0}(y)| &\leq 2|\xi(f')|^\alpha_\infty (|x - b_1|^\alpha + |y - b_2|^\alpha) \\ &\leq 2^{2-\alpha} |\xi(f')|^\alpha_\infty |x - y|^\alpha. \end{aligned}$$

Next, it is convenient to write $\eta_{k_0} = \eta_{1,k_0} + \eta_{2,k_0}$, where $\eta_{2,k_0}(x) = 0$ if $x \in p \in \overline{\mathcal{P}}_{k_0}$ and $\eta_{2,k_0}(x) = \eta_{k_0}(x)$ if $x \in p \in \widehat{\mathcal{P}}_{k_0}$.

⁷ Indeed, let $A(L) = \{x : |f'(x)| \geq L\}$, then $\lim_{L \rightarrow \infty} m(A(L)) = 0$ and

$$\left| \int h(\xi - \tilde{\xi}_\varepsilon) f' \varphi' \circ f \right| \leq C_\# \|h\|_{L^1} \sqrt{\varepsilon} + C_\# \|h\|_{L^{\frac{1}{\alpha}}} m\left(A(\varepsilon^{-\frac{1}{2}})\right)^{\frac{1}{1-\alpha}}$$

so $\lim_{\varepsilon \rightarrow 0} \int h(\xi - \tilde{\xi}_\varepsilon)(\varphi \circ f)' = 0$.

⁸ From now on we will write $(\varphi \circ f)'$ for $\sum_{p \in \mathcal{P}_k} \mathbf{1}_p(\varphi \circ f)'$, i.e. the derivative is meant in the strong sense but only where it is defined.

Putting together the above facts we have

$$(3.4) \quad \begin{aligned} |\eta_{k_0}|_\infty &\leq 2|\xi|_\infty \\ |\eta_{k_0}|_\alpha &\leq 6|\xi \cdot (f')^\alpha|_\infty. \end{aligned}$$

Note that the above estimates hold also for each η_{i,k_0} separately. Also note that

$$\begin{aligned} \left| \int_x^y \sum_{p \in \widehat{\mathcal{P}}_{k_0}} \mathbf{1}_p(\varphi \circ f \cdot \xi_{k_0})' \right| &\leq \sum_{\substack{p \in \widehat{\mathcal{P}}_{k_0} \\ p \cap [x,y] \neq \emptyset}} \sup_{z \in p} |\xi(z) f'(z)^\alpha| |p \cap [x,y]|^\alpha \\ &\leq \left[\sum_{p \in \widehat{\mathcal{P}}_{k_0}} \sup_{z \in p} |\xi(z) f'(z)^\alpha|^{\frac{1}{1-\alpha}} \right]^{1-\alpha} |x-y|^\alpha \\ &\leq C_\# |\xi f'|_\infty^{\frac{\alpha-\gamma}{1-\gamma}} \left[\sum_{\{p \in \mathcal{P} : |p| \leq 2^{-k_0}\}} \sup_{z \in p} |\xi(z) f'(z)^\gamma|^{\frac{1}{1-\gamma}} \right]^{1-\alpha} |x-y|^\alpha, \end{aligned}$$

where we have used the hypothesis $\alpha \geq \gamma$. We can then write the first term of the second line of (3.2) as

$$\begin{aligned} \sum_{p \in \mathcal{P}_{k_0}} \int_p h(\varphi \circ f \cdot \xi_{k_0})' &= \int h \eta'_{1,k_0} + \sum_{p \in \overline{\mathcal{P}}_{k_0}} \int_p h \tilde{\ell}' + \int_p h \frac{d}{dx} \int_0^x \sum_{p \in \widehat{\mathcal{P}}_{k_0}} \mathbf{1}_p(\varphi \circ f \cdot \xi_{k_0})' \\ &\leq 7|\xi \cdot (f')^\alpha|_\infty \|h\|_\alpha + C_{k_0} |h|_{L^1} \end{aligned}$$

where we have used condition (3.1) and chosen k_0 large enough.

To estimate the second term in the second line of (3.2) note that, by construction, ρ_k is zero on the elements $p \in \widehat{\mathcal{P}}_{k+1}$. We can then consider a piecewise linear approximation ℓ_k of $\varphi \circ f$ constructed by taking linear pieces on each $p \in \overline{\mathcal{P}}_{k+1}$ and such that the two functions are equal on $\partial \overline{\mathcal{P}}_{k+1}$ and define $\eta_k(x) = 0$ if $x \in p \in \widehat{\mathcal{P}}_{k+1}$ and $\eta_k = \rho_k(\varphi \circ f - \ell_k)$ otherwise. Note that $\eta_k \in \mathcal{C}^0$ and Lipschitz. Moreover, by the hypotheses on f ,⁹

$$(3.5) \quad \sup_{x \in p \in \overline{\mathcal{P}}_k} |\rho_k^\alpha \ell'_k(x)| \leq \sup_{p \in \overline{\mathcal{P}}_k} \frac{\left[\int_p |\xi f'| \right]^\alpha}{|p|} \leq C_\# 2^{(1-\alpha)k}.$$

Hence, it is natural to define

$$\psi_k(x) = \int_0^x (\varphi \circ f)' \rho_k = \int_0^x \eta'_k + \int_0^x \ell'_k \rho_k.$$

Let us estimate the $|\cdot|_\alpha$ norm term by term.¹⁰

$$\left| \int_x^y \eta'_k \right| \leq C_\# |\xi(f')^\alpha|_\infty 2^{-(1-\alpha)k} |x-y|^\alpha.$$

⁹ Remember that, by construction, $|\rho_k| \leq |\xi|$ and, for $x \in p = [a, b] \in \overline{\mathcal{P}}_k$,

$$|\ell'_k(x)| = \frac{|\varphi(f(b)) - \varphi(f(a))|}{b-a} \leq \frac{|f(b) - f(a)|^\alpha}{b-a} \leq \frac{\left| \int_p |f'| \right|^\alpha}{|p|}.$$

¹⁰ Note that $\int_0^x \eta'_k = \sum_{p \in \overline{\mathcal{P}}_{k+1}} \int_{p \cap [0,x]} \eta'_k$ and at most two of the elements of the sum are non zero, given that η_k is zero at the boundaries of \mathcal{P}_{k+1} .

Next, let $\tau \in (0, 1]$ to be chosen later. If $|x - y| \geq 2^{-\tau k}$, then¹¹

$$\begin{aligned} \left| \int_x^y \ell'_k \rho_k \right| &\leq \sum_{\substack{p \in \overline{\mathcal{P}}_k \\ p \cap [x, y] \neq \emptyset}} \int_{p \cap [x, y]} \frac{\left[\int_p |f'| \right]^\alpha}{|p|} |\rho_k| \leq \sum_{\substack{p \in \overline{\mathcal{P}}_k \\ p \cap [x, y] \neq \emptyset}} \left| \int_p |f'|^r \right|^{\frac{\alpha}{r}} 2^{-(\alpha - \frac{\alpha}{r} + \beta)k} \\ &\leq C_\# |f'|_{L^r}^\alpha |x - y|^{1 - \frac{\alpha}{r}} 2^{(1 - \alpha - \beta)k} \leq C_\# |x - y|^\alpha 2^{(1 - \alpha - \beta)k} \end{aligned}$$

provided $\alpha \leq \frac{r}{r+1}$. We have then a negative exponent if $\alpha > 1 - \beta$. If instead $\alpha > \frac{r}{r+1}$ we can end the estimate as follows

$$\left| \int_x^y \ell'_k \rho_k \right| \leq C_\# |x - y|^\alpha 2^{[(\frac{r+1}{r}\alpha - 1)\tau + 1 - \alpha - \beta]k}$$

and we obtain a negative exponent provided $\tau < \frac{\alpha + \beta - 1}{\frac{r+1}{r}\alpha - 1}$. On the other hand, if $|x - y| \leq 2^{-\tau k}$, then (remembering (3.5))

$$\left| \int_x^y \ell'_k \rho_k \right| \leq |x - y| 2^{(1 - \alpha)k} 2^{-\beta(1 - \alpha)k} \leq C_\# |x - y|^\alpha 2^{(1 - \tau - \beta)(1 - \alpha)k}.$$

The latter exponent is negative provided $\tau > 1 - \beta$. Finally, notice that if $\beta > \frac{1}{r+1}$, then $\frac{\alpha + \beta - 1}{\frac{r+1}{r}\alpha - 1} > 1 - \beta$ and hence there always exists a choice of τ that makes all the above quantities exponentially small in k . Hence, the contribution of the sum over $k \geq k_0$ can be made arbitrarily small by choosing k_0 large enough. \square

We conclude with an alternative result, in order to give a taste of the available possibilities.

Theorem 3.7. *If the potential is uniformly β -Hölder on the elements of the partition (see Remark 3.5), then we do not need to impose any condition on the integrability of f' and Theorem 3.1 holds under the single condition $1 > \alpha > \max\{\gamma, 1 - \beta\}$.*

Proof. We just need to prove Lemma 3.6 under the new condition. All the previous arguments are valid, the only difference is that now we do not need to introduce $\tilde{\xi}_\varepsilon$ and the approximation scheme yields

$$|\rho_k(x)| \leq C_\# 2^{-\beta k} |\xi(x)|$$

rather than $|\rho_k| \leq C_\# 2^{-\beta k}$ as before. Accordingly, we can easily conclude the proof of Lemma 3.6 as follows.

$$\left| \int_x^y \ell'_k \rho_k \right| \leq C_\# \int_x^y 2^{-\beta k} \frac{\left(\int_p |f' \xi| \right)^\alpha}{|p|} \leq C_\# 2^{-(\alpha + \beta - 1)k} |x - y|.$$

\square

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¹¹ Remark that $\#\{p \in \overline{\mathcal{P}}_k : p \cap [x, y] \neq \emptyset\} \leq C_\# 2^k |x - y|$ since the elements of $\overline{\mathcal{P}}_k$ have length larger than 2^{-k} .

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